Group based problems 2

1. **Semi - group :** (Already discuss in previous lecture)

2. **Sub semi-group :** Consider a semi group $(A, *)$ and $B \subseteq A$ then the system $(B, *)$ is said to be sub semi-group if the set B is closed under the operation $*$.

For example :- Consider a semi-group $(N, +)$, where N is the set of all natural numbers and $+$ is the addition operation

The algebric system $(E, +)$ is a semi-subgroup of $(N, +)$, where E is the set of all +ve integers

3. **Monoid :-** let A non-empty set G togther with a binary composition $*$ is said to be Monoid if it satisfies following properties

(1) closure property
(2) Associative property
(3) Existence of identity.

When we take defination of semi-group in that time only (1), and (2) properties are satisfied

(i) $\forall\, a, b \in G \Rightarrow a * b \in G$

(ii) $\forall\, a, b, c \in G \Rightarrow (a * b) * c = a * (b * c)$

(iii) $\forall\, a \in G,\ \exists\, e \in G$ s.t $a * e = a = e * a$.

(4) Sub monoid :- let us consider, a Monoid $(M, *)$, where $*$ is the binary operation and M is the set of all integers. Then $(M_1, *)$ is sub monoid of $(M, *)$, where M_1 is defined as $M_1 = \{a^i \mid i$ is from 0 to n, a +ve integer and $a \in M\}$

(5) Q:. In a group G, Prove that

$$(ab)^{-1} = b^{-1}a^{-1} \quad \forall\, a, b \in G$$

under the operation multiplication '.'

Sol: We know that for the existence of the inverse of any $a \in G$ s.t $a * b = e = b * a$ (under operation $*$)

or

$\underline{a.b} = \underline{e} = \underline{b.a}$ (If under operation .)

which is correct.

$(a * b)^{-1} = a^{-1} b^{-1}$ ×

$(a * b)^{-1} = a^{-1} * b^{-1}$ ×

$(a * b)^{-1} = b^{-1} * a^{-1}$

$(a * b)^{-1} = b^{-1}\, a^{-1}$ ×

Then we say that $\boxed{a^{-1} = b}$ $\boxed{(ab)^{-1} = b^{-1}a^{-1}}$

Now

$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$

$(a*b)^{-1} = b^{-1}a^{-1}$ ✓

Now

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$$

$$= a\,e\,a^{-1} \qquad \{\because bb^{-1} = e$$

$$= a\,a^{-1} \qquad \{\because a\cdot e = a$$

$$= e \qquad \{\because aa^{-1} = e$$

$$(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)\,b \qquad \{\because a^{-1}a = e$$

$$= b^{-1}(e)\,b$$

$$= b^{-1}(eb)$$

$$= b^{-1}b \qquad \{\because eb = b$$

$$= e \qquad \{\, b^{-1}b = e$$

$\therefore\ (ab)(b^{-1}a^{-1}) = e = (b^{-1}a^{-1})(ab)$

Then by the defination of inverse

$$(ab)^{-1} = b^{-1}a^{-1}$$ Proved.

Q:- let G be the set of two by two invertible matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$ belonging to G if $ad-bc \neq 0$ is a group with matrix multiplication.

Then which is true.

... is a normal subgroup of G

Then which is true.

(i) $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, a \neq 0 \right.$ is a normal subgroup of G

(ii) $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, a \neq 0 \right.$ is a subgroup of G

(iii) both (i) and (ii)

(iv) None of these.

Sol: First of all we check H is a subgroup of group G.

let $H_1 = \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix}$, $H_2 = \begin{bmatrix} a_2 & 0 \\ 0 & a_2 \end{bmatrix}$

Here $a_1 \neq 0$, $a_2 \neq 0$

be two elements of H.

$$H_1 H_2 = \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} a_1 a_2 & 0 \\ 0 & a_1 a_2 \end{bmatrix} \in H,$$

$\because a_1 \neq 0$
$a_2 \neq 0$
Then $a_1 a_2 \neq 0$

It holds for closure property.

Now we check its inverse property.

for that find out H_1^{-1},

$$\operatorname{Adj} H_1 = \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix}' = \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix}$$

$$|H_1| = \begin{vmatrix} a_1 & 0 \\ 0 & a_1 \end{vmatrix} = a_1^2$$

$$H_1^{-1} = \frac{1}{|H_1|} \operatorname{Adj}(H_1) = \frac{1}{a_1^2} \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix} = \begin{bmatrix} 1/a_1 & 0 \\ 0 & 1/a_1 \end{bmatrix} \in H$$

$$\because a_1 \neq 0$$

$\therefore$ H is closed under multiplication and each element of H has multiplicative inverse.

$\therefore$ H is subgroup of group G.

let $G_1 = \begin{pmatrix} x & y \\ z & u \end{pmatrix}$ be any element of G,

$$xu - yz \neq 0$$

$$\operatorname{Adj} G_1 = \begin{pmatrix} u & -z \\ -y & x \end{pmatrix}' = \begin{pmatrix} u & -y \\ -z & x \end{pmatrix}$$

$$G_1^{-1} = \frac{\operatorname{Adj} G_1}{|G_1|} = \frac{1}{xu - yz} \begin{bmatrix} u & -y \\ -z & x \end{bmatrix}$$

Now $G_1 H_1 G_1^{-1} = \begin{bmatrix} x & y \\ z & u \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix} \frac{1}{xu-yz} \begin{bmatrix} u & -y \\ -z & x \end{bmatrix}$

$$= \frac{1}{xu-yz} \begin{bmatrix} xa_1 & a_1 y \\ za_1 & a_1 u \end{bmatrix} \begin{bmatrix} u & -y \\ -z & x \end{bmatrix}$$

$$= \frac{1}{xu-yz} \begin{bmatrix} a_1(xu-yz) & 0 \\ 0 & a_1(xu-yz) \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \end{bmatrix} \in H_1$$

$\therefore$ $\forall$ $G_1 \in G$ such that $G_1 H_1 G_1^{-1} \in H$ for $H_1 \in H$

$\therefore$ H is normal subgroup of group of G under the multiplication.

{Note :- A subgroup H of a group G is normal iff $g^{-1} h g \in H$ for all $g \in G$ and $h \in H$.

(iii)rd option is correct.

NKJha

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